Notes on non-extremal, charged, rotating black holes in minimal $D = 5$ gauged supergravity

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Abstract

We consider the non-extremal, charged, rotating black hole solution of five dimensional minimal gauged supergravity of Cvetic, Lu and Pope [Phys. Lett. B **598** (2004) 273]. We compute the Ashtekar-Magnon-Das mass and show it agrees with the thermodynamic mass. We find a reducible Killing tensor and integrate the geodesic equation explicitly. We also compute the Euclidean action of the black hole and show it satisfies the quantum statistical relation. Further we present a Smarr relation. We end with a discussion of applications to string theory.

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1 Introduction

There has recently been renewed interest in solutions of five dimensional gravity and super-gravities, both in the ungauged and gauged case. The bosonic sector of minimal supergravity is Einstein-Maxwell with a Chern-Simons term in the former case. In the latter case a negative cosmological constant is present.

Rotating higher dimensional black holes were first discussed in the pure vacuum case in [1]. However as is well known, it is very difficult to find black holes with both rotation and charge especially in the non-extremal case. Nonetheless significant progress in the ungauged case has been made by considering compactifications of higher dimensional theories and various solution generating techniques [2]. Analogous progress has been made recently in the context of black rings. The original vacuum solutions were found in [3], and charged generalisations followed [4, 5, 6]. Although these results imply there exist black holes and black rings with the same asymptotic charges, it seems in the extremal case, the supersymmetric black ring [7] and BMPV black hole [8] do not have overlapping charges. The task of constructing non-extremal black ring solutions has been initiated in [9].

In the gauged case, the search for charged, rotating black holes has proven more difficult; however, due to the obvious implications for the AdS_5/CFT_4 correspondence, it has attracted great interest. The uncharged solution with general rotation was found in [10]; the five dimensional Reissner-Nördstrom-AdS solutions were subsequently considered in [11]. From the context of string theory, these correspond to internal rotations in the AdS_5 spacetime and black holes carrying appropriate $U(1)$ charge in the S^5 Kaluza-Klein compactification respectively. A rotating supersymmetric solution was found in [12], though it contains naked closed timelike curves. Finally, using the form of the general solution of the gauged theory found in [13], further pathology-free examples were found and it was shown that, contrary to the minimal case, these black holes *had* to rotate [14]. Remarkably, a non-extremal solution of gauged supergravity has been recently found in [15] with equal angular momenta parameters which contains the previous supersymmetric solutions as particular limits.

The authors of [16], motivated by the derivation of the general Kerr-(anti) de Sitter black holes in all dimensions [17], investigated the thermodynamic properties of these solutions. In particular they showed how one must properly define the mass and angular momenta in order to ensure the first law of thermodynamics holds. They also used standard background methods to compute the partition function rather than boundary counter-term methods, pointing out inconsistencies in the previous literature. The solution of [15] is thus interesting to study in this context because it carries electric charge in addition to rotation in two orthogonal planes. Analogous results have been obtained for the four dimensional Kerr-Newman AdS black hole in [18]. Properties of rotating, charged black holes in gauged supergravities in four, five, and seven dimensions have also been examined recently in [19].

In this article we will examine properties of this solution. In section two we review the solution and how it is parameterized and outline some basic characteristics. We follow this by looking at the conserved charges (electric charge, momentum, and mass). The mass is computed via the conformal boundary approach of Ashtekar, Magnon and Das [20] and is shown to agree with the so called 'thermodynamic mass' found in [21] which was constructed to satisfy the first law of black hole mechanics. In section four, we consider symmetries of the spacetime and identify a reducible Killing tensor which allows one to separate the Hamilton-Jacobi equation explicitly. Next we turn to the computation of the Euclidean action of the grand canonical ensemble. We use standard methods and not the counter-term techniques. We show that our action and charges satisfy the quantum statistical relation and are consistent with the first law. We also give a Smarr relation. Finally we consider applications to string theory and holography.

2 Non-extremal charged, rotating black hole

The bosonic action for $D = 5$ minimal gauged supergravity is

$$S = \frac{1}{16\pi} \int \left(\sqrt{|g|}(R - 12\lambda - F^2) - \frac{2}{3\sqrt{3}} \eta^{\mu\nu\rho\sigma\delta} F_{\mu\nu} F_{\rho\sigma} A_\delta \right) \tag{1}$$

where $\eta^{\mu\nu\rho\sigma\delta}$ is the alternating symbol and $F = dA$. The equations of motion that follow are:

$$R_{\mu\nu} = 2 \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{6} F^2 g_{\mu\nu} \right) + 4\lambda g_{\mu\nu} \tag{2}$$

$$\nabla_\mu F^{\mu\nu} = \frac{1}{2\sqrt{3}\sqrt{|g|}} \eta^{\nu\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}. \tag{3}$$

A solution to this theory was found by Cvetic, Lu and Pope [15] representing a black hole with spherical horizon topology parameterised by its mass, electric charge and one independent angular momentum. The solution they presented depends on an extra parameter. However subsequently it was shown in [21] that by performing a coordinate transformation $\psi \to \psi + (\text{const})t$ one may remove this additional hair which is thus unphysical. It is this simplified version of the metric we shall work with. It is most concisely written in terms of left invariant one forms σ_i on $SU(2)$ which satisfy $d\sigma_1 = \sigma_2 \wedge \sigma_3$ and cyclic permutations thereof. Explicitly:

$$ds^2 = -\frac{r^2 W(r)}{4b(r)^2} dt^2 + \frac{dr^2}{W(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + b(r)^2(\sigma_3 + f(r)dt)^2 \tag{4}$$

$$A = \frac{\sqrt{3}}{2} \frac{q}{r^2} \left(dt - \frac{1}{2} j\sigma_3 \right) \tag{5}$$

where the functions W, b and f are defined in the Appendix. The event horizon is located at the largest real root of $W(r)$, say $r = r_+$. Note that the absence of closed timelike curves constrains

the parameters somewhat as has already been discussed by [15, 21]. These black holes are in general non-supersymmetric though in particular limits they reduce to the Klemm-Sabra black hole ($p = 0$) [12] and the causally sound solution of Gutowski and Reall [14]. The reader is referred to [21] for the details.

3   Asymptotic charges

In this section we will compute the mass, angular momentum and charge of the black hole solution above. Since it is not asymptotically flat, the calculation of mass is non-trivial. We want to compare the gravitational mass to the 'thermodynamic' mass presented in [21]. The electric charge is defined as

$$Q = \frac{1}{4\pi} \int_{S^3} \left(*F - \frac{2}{\sqrt{3}} A \wedge F \right) \tag{6}$$

where S^3 is the 3-sphere at infinity ($r \to \infty$, $t = \text{const}$), and the Chern-Simons term is required to ensure Q is a conserved quantity. For the black hole in question one finds that the Chern-Simons term does not contribute and the result is[‡]:

$$Q = \frac{\sqrt{3}}{2} \pi q. \tag{7}$$

The angular momentum may be obtained via Komar[§] integrals and for the Killing vector $2\partial_\psi$ is given by [21]

$$J = \frac{\pi}{4} j(2p - q). \tag{8}$$

Finally the mass may be computed in a number of ways. Here we will consider the conformal mass of Ashtekar, Magnon, and Das which was successfully used in [16] to compute the energy of the uncharged Kerr-(A)dS spacetimes in all dimensions. By 'successfully' we mean that these masses coincide with the thermodynamic mass.

It is convenient to set $\lambda = -\frac{1}{\ell^2}$ to conform to standard conventions. Define the conformally rescaled metric $\bar{g}_{ab} = \Omega^2 g_{ab}$ where $\Omega = \frac{\ell}{r}$. As $r \to \infty$ one sees the boundary of the conformally rescaled metric is the Einstein static universe with metric

$$d\bar{s}^2 = -dt^2 + \ell^2 d\Omega_3^2 \tag{9}$$

which may be thought of as the conformal compactification of four dimensional Minkowski spacetime. As in the uncharged case, the conformal boundary of the black hole is $\mathbb{R} \times S^3$. Let $C^\alpha{}_{\beta\gamma\delta}$ be the Weyl tensor of either metric above. Setting $n = d\Omega$, the electric part of the Weyl tensor is

$$\bar{\mathcal{E}}^\mu{}_\nu = \frac{\ell^2}{\Omega^2} \bar{g}^{\alpha\gamma} \bar{g}^{\beta\sigma} n_\gamma n_\sigma C^\mu{}_{\alpha\nu\beta}. \tag{10}$$

[‡]In [21] it is stated that $Q = q$. Presumably they used a different normalisation.

[§]We define $J[m] = \frac{1}{16\pi} \int_{S^3} *dm$. For the case at hand $m = 2\partial_\psi$.

The conserved quantity associated with the Killing vector K is

$$Q[K] = \frac{\ell}{16\pi} \int_\Sigma \bar{\mathcal{E}}^\mu{}_\nu K^\nu d\Sigma_\mu. \tag{11}$$

Here the integral is taken over the S^3 at infinity with timelike normal dt. To compute the mass we take K to be the timelike Killing vector *non-rotating* at infinity. From the leading order term of the relevant component of the Weyl tensor

$$C^t{}_{rtr} = \frac{\ell^2(6p - 6q - 2\lambda j^2 p)}{r^6} + O(r^{-8}) \tag{12}$$

it follows immediately $\bar{\mathcal{E}}^t{}_t = \frac{6p - 6q - 2\lambda j^2 p}{\ell^4}$. The metric on the conformal boundary on constant time slices is simply that of an S^3 of radius ℓ. Inserting this into (11) yields

$$Q[\partial_t] = \frac{\pi}{4}(3p - 3q - \lambda j^2 p) \tag{13}$$

which, as we indicate in a later section, is in agreement with the thermodynamic mass. A further quantity of physical interest is the gyromagnetic ratio of these charged, rotating AdS black holes. For completeness we have computed this:

$$g = \frac{3p - 3q - \lambda j^2 p}{2p - q} \tag{14}$$

where $\mu_{ij} = g\left(\frac{Q}{2M}\right) J_{ij}$, μ_{ij} are the dipole moments and J_{ij} are the angular momenta in Cartesian coordinates. In the asymptotically flat case $3/2 \leq g \leq 3$ where the upper inequality is saturated if and only if $p = 0$ which is the BPS condition.

4 Symmetries and geodesics

The isometry group of the black hole is $\mathbb{R} \times SU(2) \times U(1)$. This is generated by the Killing vectors ∂_t, R_i and L_3 where L_i are the left invariant vector fields on $SU(2)$ dual to σ_i and R_i are the right invariant vector fields. An interesting question is whether we have enough symmetry to render geodesic motion in this spacetime Liouville integrable. To this end we first calculate the Hamiltonian of an uncharged particle in this background, $H = g^{\mu\nu} p_\mu p_\nu$. The calculation amounts to finding the inverse metric $g^{\mu\nu}$. This is most easily done in an orthonormal frame. We thus define the vielbeins ω^μ by:

$$\omega^0 = \frac{r\sqrt{W}}{2b} dt, \qquad \omega^r = \frac{dr}{\sqrt{W}}, \qquad \omega^1 = \frac{r}{2}\sigma_1, \qquad \omega^2 = \frac{r}{2}\sigma_2, \qquad \omega^3 = b(\sigma_3 + f dt) \tag{15}$$

such that $ds^2 = \eta_{\mu\nu}\omega^\mu\omega^\nu$. The dual vectors e_μ defined by $\langle\omega^\mu, e_\nu\rangle = \delta^\mu_\nu$ are:

$$e_0 = \frac{2b}{r\sqrt{W}}(\partial_t - f L_3), \qquad e_r = \sqrt{W}\partial_r \qquad e_1 = \frac{2}{r}L_1, \qquad e_2 = \frac{2}{r}L_2, \qquad e_3 = \frac{1}{b}L_3. \tag{16}$$

This allows one to easily write down the inverse metric tensor $\left(\frac{\partial}{\partial s}\right)^2 = \eta^{\mu\nu} e_\mu e_\nu$ which in our case is:

$$\left(\frac{\partial}{\partial s}\right)^2 = -\frac{4b^2}{r^2 W}\left(\partial_t - fL_3\right)^2 + W\partial_r^2 + \frac{4}{r^2}(L_1^2 + L_2^2) + \frac{1}{b^2}L_3^2. \tag{17}$$

Define the functions $E = -p_t$ and $M_i = L_i^\mu p_\mu$ and $N_i = R_i^\mu p_\mu$. Then it is clear that E, M_3 and N_3 must commute with the Hamiltonian H as they are constants along geodesics. The Hamiltonian for an uncharged particle is easy to deduce from the inverse metric:

$$H = -\frac{4b^2}{r^2 W}\left(E + fM_3\right)^2 + Wp_r^2 + \frac{4}{r^2}(M_1^2 + M_2^2) + \frac{1}{b^2}M_3^2. \tag{18}$$

It is easy to see that $K = M_1^2 + M_2^2 + M_3^2$ Poisson commutes with the Hamiltonian, as well as E, M_3 and N_3. It is actually the quadratic Casimir of the $su(2)$ algebra realised by the functions M_i under the Poisson bracket, $\{M_i, M_j\} = \epsilon_{ijk} M_k$. We have thus proved that this Hamiltonian system is Liouville integrable since we have a five dimensional configuration space and have found five Poisson commuting functions H, K, E, M_3, N_3. In fact the conserved quantity K corresponds to a reducible Killing tensor $K_{\mu\nu}$ given by $K = K^{\mu\nu} p_\mu p_\nu$ which can be read off as

$$K^{\mu\nu} = L_i^\mu L_i^\nu = R_i^\mu R_i^\nu. \tag{19}$$

From the second equality it follows that it is a reducible Killing tensor. Since the black hole considered here has equal angular momenta parameters with respect to orthogonal planes, this result is expected. Presumably a more general charged AdS black hole would possess an irreducible Killing tensor as has been shown in the uncharged case [22]. The existence of a Killing tensor is related to the separability of the Hamilton-Jacobi equation for particle motion on the space-time. In fact the HJ equation for particle motion is:

$$\frac{\partial S}{\partial l} + g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = 0 \tag{20}$$

where S is Hamilton's principal function which is a type II generating function for a canonical transformation $(x^\mu, p_\nu, H) \to (X^\mu, P_\nu, 0)$. Thus $p_\mu = \partial S/\partial x^\mu$ and $X^\mu = \partial S/\partial P_\mu$. Noting

$$(L_1 S)^2 + (L_2 S)^2 = \left(\cot\theta \partial_\psi S - \frac{1}{\sin\theta}\partial_\phi S\right)^2 + (\partial_\theta S)^2 \tag{21}$$

it becomes clear that the Hamilton-Jacobi equation is separable in the sense

$$S = m^2 l - Et + J_\psi \psi + J_\phi \phi + \Theta(\theta) + R(r), \tag{22}$$

where

$$\left(\cot\theta J_\psi - \frac{1}{\sin\theta}J_\phi\right)^2 + \Theta'(\theta)^2 = C \tag{23}$$

$$-\frac{4b^2(r)}{r^2 W(r)}\left(E + f(r)J_\psi\right)^2 + W(r)R'(r)^2 + \frac{4}{r^2}C + \frac{1}{b^2}J_\psi^2 + m^2 = 0. \tag{24}$$

The constant C is the separation constant and can easily be related to the conserved quantity K arising from the Killing tensor $K_{\mu\nu}$ which we found above. Explicitly $K = C + J_\psi^2$. Using the generating function S one may now obtain the geodesics explicitly, up to quadratures. By differentiating S with respect to $m^2, C, E, J_\psi, J_\phi$ respectively we get:

$$l = \frac{1}{2} \int dr \frac{1}{WR'(r)}, \tag{25}$$

$$\int d\theta \frac{1}{\Theta'(\theta)} = \int dr \frac{4}{r^2 W} \tag{26}$$

$$t = \int dr \frac{4b^2(E + fJ_\psi)}{r^2 W^2 R'(r)} \tag{27}$$

$$\psi = \int d\theta \frac{\cot\theta \left(\cot\theta J_\psi - \frac{J_\phi}{\sin\theta}\right)}{\Theta'(\theta)} + \int dr \left(\frac{4b^2 f(E + fJ_\psi)}{r^2 W^2 R'(r)} - \frac{J_\psi}{b^2 W R'(r)}\right) \tag{28}$$

$$\phi = -\int d\theta \frac{\left(\cot\theta J_\psi - \frac{J_\phi}{\sin\theta}\right)}{\sin\theta \Theta'(\theta)}. \tag{29}$$

One may also consider the motion of charged particles for which one needs the Hamiltonian

$$H = g^{\mu\nu}(p_\mu + eA_\mu)(p_\nu + eA_\nu) \tag{30}$$

where e is the charge of the particle. Since $\mathcal{L}_{V_i} A = 0$ for all the Killing vectors V_i of the black hole spacetime, it immediately follows that E, M_3, N_3, K still Poisson commute with H and thus this system is also Liouville integrable.

As a final remark, we should note that the Klein-Gordon equation, describing a massive scalar field in this background is also separable as a consequence of the hidden symmetry associated with the Killing tensor. The general solution looks like $\Phi = N e^{-iEt + im_\psi \psi + im_\phi \phi} P(\cos\theta) f(r)$, where $P(\cos\theta)$ can be written in terms of Jacobi polynomials.

5   Thermodynamics

The black holes studied here can be shown to satisfy the first law of black hole mechanics [21], thus suggesting that $S = A/4$. We would like to make this more concrete by computing the gravitational partition function and hence the free energy of the system to check the quantum statistical relation as well as the first law of thermodynamics. Such a program has been carried out for the general Kerr-AdS spacetimes in all dimensions in [16].

We begin by briefly reviewing the thermodynamic quantities for completeness. The Killing vector $\xi = \partial_t + \Omega_H 2\partial_\psi$ is null on the horizon where

$$\Omega_H = -f(r_+)/2. \tag{31}$$

The potential measured in a co-rotating frame on the horizon, defined by $\phi_H = \xi^\mu A_\mu$, is

$$\Phi_H = \frac{\sqrt{3}q}{2r_+^2}(1 - j\Omega_H). \tag{32}$$

The area of the horizon is easily computed to be

$$A = 4\pi^2 r_+^2 b(r_+) \tag{33}$$

as is the surface gravity κ which we find to be

$$\kappa = \frac{r_+ W'(r_+)}{4b(r_+)}. \tag{34}$$

Armed with these quantities one may verify that

$$dM = \frac{\kappa}{8\pi}dA + 2\Omega_H dJ + \Phi_H dQ \tag{35}$$

which defines the thermodynamic mass M to be

$$M = \frac{\pi}{4}(3p - 3q - \lambda j^2 p). \tag{36}$$

This is a tedious task for which one needs to use the fact that $W(r_+) = 0$. We should pause to emphasise the non-triviality of this statement. We can only define such a thermodynamic mass because the RHS of (35) is an exact differential. Also, we should emphasise that $M = Q[\partial_t]$ which is the statement that the mass (13) computed from the prescription of Ashtekar, Magnon, and Das agrees with the thermodynamic mass.

To compute the partition function we will follow the standard prescription [23] and carry out the path integral along a complex contour and then analytically continue back. A real Euclidean section of the charged, rotating, asymptotically AdS black hole can be found upon making the substitutions $t = -i\tau$ and $j = i\hat{j}$. Note that the charge parameter q cannot be so analytically continued and so the Maxwell field is pure imaginary on the Euclidean section. For the following computation we will henceforth work with this Euclidean section. Let the largest positive root of $W(r)$ be r_+. Removal of the conical singularity at $r = r_+$ requires that τ be identified with period

$$\beta = \frac{1}{T} = \frac{8\pi b(r_+)}{r_+ W'(r+)} \tag{37}$$

thus confirming the standard relation $T = \kappa/2\pi$.

The on-shell Euclidean action is

$$\hat{I} = -\frac{1}{2\pi}\int_M \sqrt{g}\lambda + \frac{1}{12\pi}\int_{\partial M} \sqrt{h}F^{\mu\nu}A_\nu n_\mu - \frac{1}{8\pi}\int_{\partial M}\sqrt{h}K \tag{38}$$

where we have included the usual Gibbons-Hawking boundary term. The Euclidean action $\hat{I}$ is defined via $I = i\hat{I}$. Here the boundary of the spacetime M is taken to be a hypersurface

$r = $ const with the limit $r \to \infty$ taken after performing the integral. The bulk term is obviously divergent, as in the case for asymptotically AdS spacetimes.

We begin with the second term, the contribution from the Maxwell field. Firstly, since the Killing vector ξ has zero norm on the horizon we require $\xi \cdot A = 0$ on the horizon. This is achieved upon performing the gauge transformation $A \to A - \Phi_H dt \equiv \tilde{A}$. It is easiest to work in the vielbein basis. Explicitly

$$\tilde{A} = \frac{2b}{r\sqrt{W}} \frac{\sqrt{3}q}{2r^2} \left(1 + \frac{fj}{2}\right) \omega^0 - \frac{\sqrt{3}qj}{4br^2}\omega^3 - A_H \tag{39}$$

where we write $A_H = \Phi_H dt$. Note that $\sqrt{h} = \frac{\sqrt{W}r^3 \sin\theta}{8}$. Now, since $n^\mu = e_r^\mu$, it is clear that the only relevant components of the field strength are $F_{\hat{r}\hat{0}}$ and $F_{\hat{r}\hat{3}}$. It is easy to compute F and we find:

$$F = -\frac{2\sqrt{W}}{r}\omega^r \wedge A - \frac{\sqrt{3}qj}{4r^2}\sigma_1 \wedge \sigma_2 \tag{40}$$

from which we extract $F_{\hat{r}\hat{0}}$ and $F_{\hat{r}\hat{3}}$. By studying the asymptotics (for large r) of these components of F, A and $\sqrt{h}$ one may easily see that the only non-zero contribution to the quantity $\sqrt{h}F_{\mu\nu}\tilde{A}^\nu n^\mu$ is from $-\sqrt{h}F_{\hat{r}\hat{0}}A_H^{\hat{0}}$. We get

$$\sqrt{h}F_{\mu\nu}\tilde{A}^\nu n^\mu \sim -\frac{3}{16}\sin\theta\frac{q^2}{r_+^2}(1 - j\Omega_H) \tag{41}$$

which allows us to get

$$\frac{1}{12\pi}\int_{\partial M}\sqrt{h}F_{\mu\nu}\tilde{A}^\nu n^\mu = -\frac{\pi\beta q^2}{4r_+^2}(1 - j\Omega_H). \tag{42}$$

Now we turn to the calculation of the first term in (38), which is somewhat more subtle. The reason is that as it stands it is divergent due to AdS having infinite volume, and hence we need to regularize it. The method consists of performing the integral over the r coordinate only up to some finite, yet large, value which we shall denote by R. Then we subtract the same integral evaluated for the background AdS metric which we may get from the black hole by setting $p = q = 0$. However there is a subtlety. The black hole metric and the metric for AdS must match at $r = R$ for large R. This can be taken care of by rescaling the τ coordinate such that:

$$(1 - \lambda R^2)\tau_0^2 = W(R)\tau^2. \tag{43}$$

This can be thought of as a rescaling of the temperature

$$\beta_0 = \sqrt{\frac{W(R)}{1 - \lambda R^2}}\beta. \tag{44}$$

The final piece of information we need is $\sqrt{g} = r^3 \sin\theta/8$. It is then easy to calculate the integrals

$$\int_{r_+\leq r\leq R}\sqrt{g} - \int_{0\leq r\leq R}\sqrt{\bar{g}} = \frac{\pi^2}{2}(R^4 - r_+^4)(\beta - \beta_0) - \frac{\pi^2}{2}r_+^4\beta_0 \tag{45}$$

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where $\bar{g}$ is the AdS metric with the τ coordinate rescaled as discussed above. Next we need to find the limit of this expression as $R \to \infty$. To do this note that

$$\frac{\beta - \beta_0}{\beta} = -\frac{p(1 + \lambda j^2) - q}{\lambda R^4} + O(R^{-6}) \tag{46}$$

which allows one to deduce that

$$-\frac{\lambda}{2\pi} \left(\int_{r_+ \leq r \leq R} \sqrt{g} - \int_{0 \leq r \leq R} \sqrt{\bar{g}} \right) \to \frac{\pi\beta(p(1 + \lambda j^2) - q)}{4} + \frac{\pi}{4}\lambda r_+^4 \beta \tag{47}$$

as $R \to \infty$. This is the regularised contribution to the action from the bulk integral.

Finally we turn to the last surface term involving the extrinsic curvature. This is facilitated by using the identity $\mathcal{L}_n \sqrt{h} = \sqrt{h}K$. This integral will also have to be regularised by the subtraction method used for the bulk integral. It is then straightforward to show

$$\int_{r=R} \sqrt{h} - \int_{r=R} \sqrt{\bar{h}} = \text{const} + O(R^{-2}) \tag{48}$$

and thus

$$\int_{r=R} \sqrt{h}K - \int_{r=R} \sqrt{\bar{h}}\bar{K} = \sqrt{W(R)}\frac{\partial}{\partial R} \left(\int_{r=R} \sqrt{h} - \int_{r=R} \sqrt{\bar{h}} \right) = O(R^{-2}) \tag{49}$$

Hence, sending $R \to \infty$ shows that the contribution from the Gibbons-Hawking term to the gravitational action vanishes in our case, just like in the uncharged case. Collecting all these results gives us the Euclidean action for the black hole

$$\hat{I} = \frac{\pi\beta(p(1 + \lambda j^2) - q)}{4} - \frac{\pi\beta q^2}{4r_+^2}(1 - j\Omega_H) + \frac{\pi}{4}\lambda r_+^4 \beta. \tag{50}$$

Note that in the uncharged case ($q = 0$) this agrees with the results of [10, 16] when the rotation parameters are equal. To see this one needs to know that $p = m/\Xi^3$, $j = a$ and $r^2 \to (r^2 + a^2)/\Xi$ with of course $\lambda = -1/l^2$ and $\Xi = 1 - a^2/l^2$.

We now turn to the verification of the quantum statistical relation [16]. In a thermal ensemble of fixed temperature, angular velocity, and potential, the partition function satisfies $Z = e^{-\beta\Phi}$ where Φ is the Gibbs free energy. From the Euclidean quantum gravity point of view, in the semiclassical limit we expect

$$\beta\Phi = \hat{I} \tag{51}$$

to hold. As first observed in [23] this relation involves Planck's constant and is quantum mechanical in origin. Using the fact that $W(r_+) = 0$, it is a simple matter of tedious algebra to check that

$$T\hat{I} = M - TS - 2\Omega_H J - \Phi_H Q \tag{52}$$

if and only if $S = A/4$. This can be thought of in two ways. Either a proof of the fact that $S = A/4$ for this black hole given (51) or alternatively one may suppose $S = A/4$ and deduce the quantum statistical relation (52) above. The former viewpoint seems to be the more logical standpoint. In fact ideally one should express the partition function in terms of the intensive variables $(\beta, \Omega_H, \Phi_H)$, from which one can then derive the extensive quantities (S, J, Q). This appears to be rather difficult.

When $\lambda = 0$ one may verify the Smarr relation

$$M = \frac{3}{2}TS + \frac{3}{2}2\Omega_H J + \Phi_H Q \tag{53}$$

using the fact that $W(r_+) = 0$. When $\lambda \neq 0$ this relation does not hold. This can be thought of as due to having an extra dimensionful quantity, λ, which breaks the scale invariance of the space-time. In fact, if one treats λ as one of the extensive variables of our thermodynamic system, one can write down a Smarr relation:

$$M = \frac{3}{2}TS + \frac{3}{2}2\Omega_H J + \Phi_H Q - \Theta\lambda \tag{54}$$

where the intensive variable Θ conjugate to λ is given by:

$$\Theta = -\frac{\pi}{32r_+^4 b(r_+)^2}(3r_+^{10} + 10pj^2 r_+^6 + 8p^2 j^4 r_+^2 - 3j^2 q^2 r_+^4 - 4j^4 q^2 p). \tag{55}$$

From this one may now deduce a generalised first law using Euler's theorem for homogeneous functions. To do this we note that $M = M(S, Q, J, \lambda)$ and that the dimensions, in units of length are $[M] = 2$, $[Q] = 2$, $[J] = 3$, $[S] = 3$, $[\lambda] = -2$. Then application of Euler's theorem tells us

$$M = \frac{3}{2}\frac{\partial M}{\partial S}S + \frac{3}{2}\frac{\partial M}{\partial J}J + \frac{\partial M}{\partial Q}Q - \frac{\partial M}{\partial \lambda}\lambda. \tag{56}$$

From this it follows that:

$$dM = TdS + 2\Omega_H dJ + \Phi_H dQ + \Theta d\lambda. \tag{57}$$

6   Discussion

We have been primarily concerned with properties of the solution from the five dimensional standpoint. We conclude with some remarks on some applications. Solutions of minimal $N = 1$ gauged supergravity may be uplifted into Type IIB supergravity by a straightforward oxidation method. One can hence construct a general non-supersymmetric configuration from the black holes studied here. More precisely if ds_5^2 represents the five dimensional metric (4) then the ten-dimensional solution reads

$$ds_{10}^2 = ds_5^2 + \ell^2 \sum_{i=1}^{3}\left(d\mu_i^2 + \mu_i^2\left(d\xi_i + \frac{2}{\ell\sqrt{3}}A\right)^2\right). \tag{58}$$

Here $\sum_i^3 \mu_i^2 = 1$ and $0 \le \xi_i < 2\pi$ parameterise S^5. The RR five form flux is

$$F_{[5]} = (1 + \star_{10}) \left(-\frac{4}{\ell}\text{vol}_5 + \frac{\ell^2}{\sqrt{3}} \sum_{i=1}^3 \text{d}(\mu_i^2) \wedge \text{d}\xi_i \wedge \star_5 \text{F}_{[2]} \right). \tag{59}$$

Here vol_5 is the volume form on (4). These solutions are simply non extremal generalisations of the supersymmetric solutions preserving 2 supersymmetries found in [24].

From the string point of view, it is natural to consider these charged, rotating black holes solutions in the context of the AdS_5/CFT_4 correspondence. Alternatively, one can take holography at face value and try to match properties of the bulk spacetime with a particular conformal field theory living on its boundary. This has been done in the singly rotating case [25] and the static charged case [11] in the high temperature limit. It is in this limit that the supergravity action, corresponding to the strongly coupled region of the gauge theory, is found to match to the weakly coupled field theory action evaluated on the boundary up to the usual $\frac{4}{3}$ factor. Other limits that have been explored include $\Omega \to 1$ [10].

In this charged rotating case one expects similar results. As usual take $\mathcal{N} = 4$, $SU(N)$ super Yang-Mills on S^3 of radius R. Note the boundary of the black hole metric is *not* rotating. The 'charge' on the field theory side originates from the various fields carrying equal R-charges in the three $U(1)$ Cartan subgroups of the $SO(6)$ (see [26] for a more general treatment). One can show in the high temperature limit that the contribution to the partition function from the charge is sub-leading with respect to the angular momentum. Explicitly we have

$$\begin{aligned} F_{CFT}(T, \Omega, \Phi) \quad &= T_{CFT} \sum_i \int_0^\infty dl_i \int_{-l_i}^{l_i} dm_L^i \int_{-l_i}^{l_i} dm_R^i \sum_{q_i} \log(1 - \eta_i e^{-\beta(l_i/R + 2\Omega m_L^i + q_i \Phi)}) \\ &\sim -\frac{\pi^2}{6} \frac{N^2 V T_{CFT}^4}{(4R^2\Omega^2 - 1)^2} \end{aligned} \tag{60}$$

where the sum over i denotes summing over the different fields in the CFT, η_i is $+1(-1)$ for a boson (fermion), m_L, m_R and l are the quantum numbers of the $SU(2)_L \times SU(2)_R$ symmetry and q the quantum numbers of the $U(1)$ R-charge. On the strongly coupled side, it is more subtle to see how the high temperature limit should be taken, given the difficulty in writing the action in terms of the intensive variables. It would be interesting to study this limit, as well as the thermodynamic stability of these solutions.

Finally, one could extend the results of this paper to the $N = 2$, $U(1)^3$, gauged supergravity where similar non-extremal black holes were found [27].

A   Useful formulae

First we define the various functions appearing in the metric. Firstly we have

$$W(r) = 1 - 4\lambda b(r)^2 - \frac{1}{r^2}(2p - 2q) + \frac{1}{r^4}(q^2 + 2pj^2) \tag{61}$$

where the function $b(r)$ is given by

$$b(r)^2 = \frac{r^2}{4}\left(1 - \frac{j^2 q^2}{r^6} + \frac{2j^2 p}{r^4}\right). \tag{62}$$

The function $f(r)$ is given by

$$f(r) = -\frac{j}{2b(r)^2}\left(\frac{2p - q}{r^2} - \frac{q^2}{r^4}\right). \tag{63}$$

We have also made use of the left invariant 1-forms on $SU(2)$, denoted by σ_i where $i = 1, 2, 3$. They are defined by:

$$\sigma_1 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi \tag{64}$$

$$\sigma_2 = \cos\psi d\theta + \sin\psi \sin\theta d\phi \tag{65}$$

$$\sigma_3 = d\psi + \cos\theta d\phi \tag{66}$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. The left invariant vector fields L_i which form a dual basis to σ_i, so $\langle\sigma_i, L_j\rangle = \delta_{ij}$ are:

$$L_1 = -\cot\theta\cos\psi\partial_\psi - \sin\psi\partial_\theta + \frac{\cos\psi}{\sin\theta}\partial_\phi \tag{67}$$

$$L_2 = -\cot\theta\sin\psi\partial_\psi + \cos\psi\partial_\theta + \frac{\sin\psi}{\sin\theta}\partial_\phi \tag{68}$$

$$L_3 = \partial_\psi. \tag{69}$$

The right invariant vector fields are denoted by R_i, and are given by:

$$R_1 = \cot\theta\cos\phi\partial_\phi + \sin\phi\partial_\theta - \frac{\cos\phi}{\sin\theta}\partial_\psi \tag{70}$$

$$R_2 = -\cot\theta\sin\phi\partial_\phi + \cos\phi\partial_\theta + \frac{\sin\phi}{\sin\theta}\partial_\psi \tag{71}$$

$$R_3 = \partial_\phi. \tag{72}$$

References

[1] R. C. Myers and M. J. Perry, "Black Holes In Higher Dimensional Space-Times," Annals Phys. **172**, 304 (1986).

[2] M. Cvetic and D. Youm, "General Rotating Five Dimensional Black Holes of Toroidally Compactified Heterotic String," Nucl. Phys. B **476** (1996) 118, `hep-th/9603100`.

[3] R. Emparan and H. S. Reall, "A rotating black ring in five dimensions," Phys. Rev. Lett. **88**, 101101 (2002), `hep-th/0110260`.

[4] H. Elvang and R. Emparan, "Black rings, supertubes, and a stringy resolution of black hole non-uniqueness," JHEP **0311** (2003) 035, `hep-th/0310008`.

[5] H. K. Kunduri and J. Lucietti, "Electrically charged dilatonic black rings," Phys. Lett. B **609**, 143 (2005), `hep-th/0412153`.

[6] R. Emparan, "Rotating circular strings, and infinite non-uniqueness of black rings," JHEP **0403**, 064 (2004), `hep-th/0402149`.

[7] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, "Supersymmetric black rings and three-charge supertubes," Phys. Rev. D **71**, 024033 (2005), `hep-th/0408120`.

[8] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, "D-branes and spinning black holes," Phys. Lett. B **391**, 93 (1997), `hep-th/9602065`.

[9] H. Elvang, R. Emparan and P. Figueras, "Non-supersymmetric black rings as thermally excited supertubes," JHEP **0502**, 031 (2005), `hep-th/0412130`.

[10] S. W. Hawking, C. J. Hunter and M. M. Taylor-Robinson, "Rotation and the AdS/CFT correspondence", Phys. Rev. D **59** (1999) 064005, `hep-th/9811056`.

[11] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, Charged AdS black holes and catastrophic holography, Phys. Rev. D **60**, 064018 (1999), `hep-th/9902170`.

[12] D. Klemm and W. A. Sabra, "Charged rotating black holes in 5d Einstein-Maxwell-(A)dS gravity," Phys. Lett. B **503**, 147 (2001), `hep-th/0010200`.

[13] J. P. Gauntlett and J. B. Gutowski, "All supersymmetric solutions of minimal gauged supergravity in five dimensions," Phys. Rev. D **68**, 105009 (2003) [Erratum-ibid. D **70**, 089901 (2004)], `hep-th/0304064`.

[14] J. B. Gutowski and H. S. Reall, "Supersymmetric AdS(5) black holes," JHEP **0402**, 006 (2004), `hep-th/0401042`.

[15] M. Cvetic, H. Lu and C. N. Pope, "Charged Kerr-de Sitter black holes in five dimensions," Phys. Lett. B **598** (2004) 273, `hep-th/0406196`.

[16] G. W. Gibbons, M. J. Perry and C. N. Pope, "The first law of thermodynamics for Kerr - anti-de Sitter black holes" `hep-th/0408217`.

[17] G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, "Rotating black holes in higher dimensions with a cosmological constant," Phys. Rev. Lett. **93** (2004) 171102, `hep-th/0409155`.

[18] M. M. Caldarelli, G. Cognola and D. Klemm, "Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories," Class. Quant. Grav. **17** (2000) 399, `hep-th/9908022`.

[19] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, "Rotating Black Holes in Gauged Supergravities: Thermodynamics, Supersymmetric Limits, Topological Solitons and Time Machines,", `hep-th/0504080`.

[20] A. Ashtekar and S. Das, "Asymptotically anti-de Sitter space-times: Conserved quantities," Class. Quant. Grav. **17**, L17 (2000), `hep-th/9911230`.

[21] O. Madden and S. F. Ross, "On uniqueness of charged Kerr-AdS black holes in five dimensions" Class. Quant. Grav. **22** (2005) 515, `hep-th/0409188`.

[22] H. K. Kunduri and J. Lucietti, "Integrability and the Kerr-(A)dS black hole in five dimensions", `hep-th/0502124`.

[23] G. W. Gibbons and S. W. Hawking, "Action Integrals And Partition Functions In Quantum Gravity," Phys. Rev. D **15** (1977) 2752.

[24] J. P. Gauntlett, J. B. Gutowski and N. V. Suryanarayana, "A deformation of $AdS_5 \times S^5$," Class. Quant. Grav. **21** (2004) 5021, `hep-th/0406188`.

[25] D. S. Berman and M. K. Parikh, "Holography and rotating AdS black holes", Phys. Lett. B **463** (1999) 168, `hep-th/9907003`.

[26] T. Harmark and N. A. Obers, "Thermodynamics of spinning branes and their dual field theories", JHEP **0001** (2000) 008, `hep-th/9910036`.

[27] M. Cvetic, H. Lu and C. N. Pope, "Charged rotating black holes in five dimensional $U(1)^3$ gauged N = 2 supergravity,", Phys. Rev. D **70** (2004) 081502 `hep-th/0407058`.